Theorem:

Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point x of (a, b). More precisely

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-)$$

Proof:

Let f be monotonically increasing function on (a, b)

$$\Longrightarrow f(t) \leq f(x) \quad \forall\, t \in (a, x)$$

$\Longrightarrow f(x)$ is an upper bound of $\{f(t) | a < t < x\}$

$\Longrightarrow \sup_{a<t<x} f(t)$ exists and $\sup_{a<t<x} f(t) \leq f(x)$

Let $A = \sup_{a<t<x} f(t)$

Claim: $A = f(x-)$

Let $\epsilon > 0$ be given

Since $A = \sup_{a<t<x} f(t)$,there exists $\delta > 0$ such that $a < x - \delta < x$ and

$$A - \epsilon < f(x - \delta) \leq A \dashrightarrow (1)$$

f is a monotonic increasing on (a, b) and $x - \delta < t < x$

$$\Longrightarrow f(x - \delta) \leq f(t) \leq A \dashrightarrow (2)$$

From(1) & (2)

$$A - \epsilon < f(x - \delta) \leq f(t) \leq A < A + \epsilon \quad (a < x - \delta < t < x)$$

$$\Longrightarrow A - \epsilon < f(t) < A + \epsilon \qquad (x - \delta < t < x)$$

$$\Longrightarrow -\epsilon < f(t) - A < \epsilon \qquad (x - \delta < t < x)$$

$$\Longrightarrow |f(t) - A| < \epsilon \qquad (x - \delta < t < x)$$

$$\Longrightarrow f(x - 1) = A$$

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \dashrightarrow (3)$$

III^{ly} we can prove that

$$f(x+) = \inf_{x<t<b} f(t) \dashrightarrow (4)$$

From(3) , (4) and $f(x) \leq f(x+)$

$\therefore \sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) \leq \inf_{x<t<b} f(t)$

Let $a < x < y < b$

$$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t)$$

Also $f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t)$

We know that,

$$\inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t)$$

$$\Longrightarrow f(x+) \leq f(y-)$$

Hence proved.

Theorem:

If f is monotonic on $[a, b]$. Then the set of discontinuities of f is countable.

Proof:

Let f be a monotonic increasing function on $[a, b]$ and E be the set of all discontinuities of f on (a, b).

Let $s_m = \left\{x_k \in (a, b) \middle| f(x_k+) - f(x_k-) \geq \frac{1}{m}\right\} \quad \forall\, m \in N$

Clearly

$$E = \cup_{m=1}^{\infty} s_m \dashrightarrow (1)$$

Let $x_1 < x_2 < \cdots < x_{n-1}$ be in s_m.

By theorem,

$$\sum_{k=1}^{n-1} [f(x_k+) - f(x_k-)] \leq f(b) - f(a)$$

$$\Longrightarrow \sum_{k=1}^{n-1} \frac{1}{m} \leq f(b) - f(a)$$

$$\Longrightarrow \frac{n-1}{m} \leq f(b) - f(a)$$

Since $f(b) - f(a)$ is always finite,

s_m must contain only finite no of elements of (a, b)

$\therefore E$ is the countable union of finite set $\quad\quad \left(By(1)\right)$

$\Longrightarrow E$ is countable.

$\Longrightarrow$ the set of discontinuities of f on $[a, b]$ is countable.

III^{ly} we can prove the result if f is monotonically decreasing on $[a, b]$

Hence proved.